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Non-linear characteristics of engineering soils

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The paper deals with the non-linear stress–strain characteristics of engineering soils and investigates some of the more important factors relating to the elasto-plastic behaviour. A review of previous literature describes some of the work already performed in this area of study, and discusses the applicability of the theories proposed, most of which have been established using results obtained from a standard tri-axial test. A non-linear stress–strain theory is formulated, using results from a "true" tri-axial test, which also takes into account the behaviour of the soil during unloading and subsequent reloading. This behaviour is shown to depend on the stress levels attained in the unloading/reloading hysteresis loop in which the reloading portion of the loop is closely linearly elastic. Finally, the suitability of the proposed equations in typical engineering calculations is discussed and a worked example is included which demonstrates how the proposed incremental stress–strain equations may be applied in analysis.

List of symbols

The notation used in the theoretical work is as follows:

E = modulus of elasticity of soil.
H = vertical depth of cavity.
K_0 = coefficient of earth pressure at rest assumed constant with depth.
K_r = Rankine ratio = $(1 - \sin\phi)/(1 + \sin\phi)$.
$K, K_1, K_2, f_1, f_2, g, m, n$ = constants found experimentally.
k = R_0^2/R_1^2.
p_h = stress in any horizontal direction when $p_x = p_z = p_h$.
p_r = radial stress.
p_x = stress in x-direction.
p_y = vertical stress.
p_z = stress in z-direction.
p_{xi}, p_{yi}, p_{zi} = stress levels in co-ordinate directions at commencement of unloading/reloading cycle.
p_{xf}, p_{yf}, p_{zf} = stress levels in co-ordinate directions at end of unloading portion of unloading/reloading cycle.
p_θ = circumferential stress.
p_u = ultimate pressure in cavity.
R_0 = initial radius of cavity.
R_1 = ultimate radius of cavity.
R_2 = radius of plastic zone.
r = a radial distance from centre of cavity to typical element in elastic or plastic zone.
r_1, r_2, r_3 = stress ratios = p_x (ultimate)$/p_{xi}$, p_y (ultimate)$/p_{yi}$, p_z (ultimate)$/p_{zi}$ respectively.
t = step length.
w = radial displacement of elastic zone.
γ = soil density.
δ = average volumetric strain of plastic zone.
δ_r = volumetric strain of increment of plastic zone.
$\delta p_r, \delta\epsilon_r$, etc. = a small increment of the quantity concerned.
ϵ_r = radial strain.
ϵ_x = strain in x-direction.
ϵ_y = vertical strain.
ϵ_z = strain in z-direction.
ϵ_θ = circumferential strain.
ϵ_{xyi}, etc = strain in the x-direction due to loading in y-direction at commencement of unloading/reloading cycle, etc.
ϵ_{xyf}, etc = strain in the x-direction due to loading in y-direction at end of unloading portion of unloading/reloading cycle, etc.
ν = Poisson's ratio.
ϕ = angle of internal friction of soil.

1. Introduction

The term soil, in an engineering sense, is used to describe the sediments and deposits of solid particles produced by the disintegration of rock. Such soils, therefore, consist of solid particles which are assembled in relatively open packing, and variations in the grading and packing of these particles can produce samples which may be extremely dissimilar in both composition and behaviour. It should be emphasized that although the approach in this work tends to discuss the behaviour of a particulate solid material in terms of an "engineering soil", a formulation of the response of such a material under a varying stress field may be fundamental to other areas of study such as those involving packing and pressing processes.

It is perhaps understandable that early analyses of stress and strain in soils tended to assume that behaviour was that of a perfectly elastic solid with a constant modulus of elasticity and Poisson's ratio. There are a multiplicity of factors influencing soil behaviour, the relative assessment of which was not immediately apparent, and although it was generally acknowledged that it was unwise to place too much reliance on linear elastic theory [1], few other solutions were available; and those that were available were relatively intractable due to lack of advanced computational facilities. Furthermore, elastic idealization has been given impetus due to the fact that there are certain classes of problems which are satisfactorily described by this approach.

Within the last decade or so researchers have displayed increasing interest in the factors governing the non-linear stress–strain behaviour of soils. Besides soil structure these include such factors as water content and drainage conditions, type and duration of loading and stress history. The latter items have attracted considerable interest because the behaviour of soils are greatly influenced by the anisotropy which is induced by the loading system. Moreover, it has been observed [2] that successive loading, unloading and reloading of soil samples produces approximately linear elastic behaviour with only small hysteresis of the soil during the unloading and reloading cycle. It would be important that any elasto-plastic representation of a soil should be able to describe this facet of non-linear behaviour satisfactorily. In connection with this factor it is of interest that under certain circumstances a naturally occurring granular soil may show, within a given range of loading/unloading, a linearly elastic response. A further discussion of this feature occurs later.

Until recently, with one or two exceptions [3], investigation into soil behaviour has been conducted mainly using a standard tri-axial apparatus in which the intermediate and minor principal stresses are equal. This may be satisfactory for producing simplified stress–strain relationships and, indeed, has been used fairly recently [4] for finding stress-dependent coefficients under these simplified conditions. However, for cases in which three-dimensional stresses and strains are involved it is important that "true" tri-axial tests are carried out, i.e. tests in which the three principal stresses may be independently varied; and proposed theoretical predictions employ expressions that involve all three principal stresses and strains. Under such circumstances it is apparent that the concepts of modulus of elasticity and Poisson's ratio are abandoned, for even in simplified stress systems these have been shown to be stress dependent. Instead, for the general case, it is clear that nine independent stress-dependent coefficients are involved in the stress–strain relationships for soils. In addition to this, many calculations in engineering soils are not properly described by the relationships derived from the simplified tri-axial procedure as the failure criteria as well as stress–strain relationships should include the effect of the intermediate principle stress.

Summarizing, therefore, engineering calculations for soils should not only include the important concepts of non-linearity, inelasticity, and stress-dependency but also all three principal stresses; and it is the purpose of this study to put forward a basic and easily worked formulation that will enable workers to satisfactorily describe soil behaviour when subjected to general loading cycles.

2. Previous work

The following non-linear stress–strain relationship of hyperbolic form which is applicable to both sands and clays has been proposed by Kondner and others [5, 6]:

$$p_y - p_{\mathrm{h}} = \frac{\epsilon_y}{a + b\epsilon_y} \tag{1}$$

where p_y and p_{h} are the major and minor (also intermediate) principal stresses respectively. ϵ_y is the strain in the principal direction and a, b are constants which may be determined experimentally. These have the following physical significance:

at $\epsilon_y = 0$; $\dfrac{d(p_y - p_h)}{d\epsilon_y} = 1/a$ = initial tangent modulus and at $\epsilon_y = \infty$; $(p_y - p_h)_{ultimate} = 1/b$ = failure asymptote.

It is a point of interest, to which attention will later be drawn, that according to the above relationship, $\epsilon_y = 0$ when $p_y = p_h$; and this is, of course, not necessarily correct. However, it must be admitted that the condition of zero stress and strain is not really applicable in practice and, therefore, the condition of zero strain may be arbitrarily applied to an already stressed soil.

Equation 1 can be rewritten in the following form:

$$\frac{\epsilon_y}{(p_y - p_h)} = a + b\epsilon_y$$

which, when plotted on transformed axes, yields a straight line relationship. It is apparent that the value of the initial tangent modulus is stress dependent, and has been shown by Janbu [7] to be a function of the minor principal stress p_h. Other researchers such as Hansen [8], Desai [9], Breth *et al.* [4] have also proposed stress–strain-relationships, but all these proposed relationships have been based on data obtained from standard tri-axial tests in which the intermediate principal stress is equal to the minor principal stress. However, Daniel [3] has, by conducting a "true" tri-axial test, produced generalized stress–strain relationships of the following type which are of an incremental nature, and applicable to dry granular media subjected to first loading:

$$\delta\epsilon_x = \frac{\delta p_x}{K\left[p_x - \dfrac{1}{K_r}\sqrt{(p_y p_z)}\right]} - \left(2 - \frac{p_x}{p_z}\right)\left[\frac{m}{\dfrac{p_x}{K_r} - p_y} - \frac{n}{\dfrac{p_x}{K_r}}\right]\delta p_y - \left(2 - \frac{p_x}{p_y}\right)\left[\frac{m}{\left(\dfrac{p_x}{K_r} - p_z\right)} - \frac{n}{\dfrac{p_x}{K_r}}\right]\delta p_z \quad (2a)$$

$$\delta\epsilon_y = -\frac{p_z}{p_y}\left[\frac{m}{\left(\dfrac{p_z}{K_r} - p_x\right)} - \frac{n}{\dfrac{p_z}{K_r}}\right]\delta p_x + \frac{\delta p_y}{K\left(p_y - \dfrac{1}{K_r}\sqrt{(p_x p_z)}\right)} - \frac{p_x}{p_y}\left[\frac{m}{\left(\dfrac{p_x}{K_r} - p_z\right)} - \frac{n}{\dfrac{p_x}{K_r}}\right]\delta p_z \quad (2b)$$

$$\delta\epsilon_z = -\left(2 - \frac{p_z}{p_y}\right)\left[\frac{m}{\left(\dfrac{p_z}{K_r} - p_x\right)} - \frac{n}{\dfrac{p_z}{K_r}}\right]\delta p_x - \frac{p_x}{p_z}\left[\frac{m}{\left(\dfrac{p_x}{K_r} - p_y\right)} - \frac{n}{\dfrac{p_x}{K_r}}\right]\delta p_y + \frac{\delta p_z}{K\left(p_z - \dfrac{1}{K_r}\sqrt{(p_x p_y)}\right)} \quad (2c)$$

where p_x, ϵ_x, etc = stresses and strains respectively in co-ordinate directions; δp_x, $\delta\epsilon_x$, etc = a small increment of the quantity concerned; K_r = Rankine ratio, K, m and n = constants found experimentally. Examination of the equations shows that the incremental strains for each increment of stress are related by stress-dependent coefficients which contain experimental parameters depending only on the type of sand.

Using Equation 2b and putting $p_x = p_z = p_h$ = constant, for a particular experiment, produces a relationship close to that represented by Equation 1 except that a zero correction has to be made because Kondner's equation does not allow for other than hydrostatic stress and zero strain occurring simultaneously. In fact, Kondner's equation is rather inadequate in the small strain range. However, this shows that in the simplified case Daniel's equations give similar results to Kondner's.

As stated in the introduction, it is important to be able to predict the behaviour of soil, not only under a gradually increasing stress field, but also during unloading and reloading cycles. Ko and Scott [10] and Holubec [11] have shown that soil exhibits an elasto-plastic behaviour such that strains induced under loading are not completely recoverable when unloading takes place. They further showed that under simplified stress conditions during the reloading cycle, nearly linear elastic behaviour occurred until the original state of loading was attained. Duncan and Chang [2] carried out some additional tests which confirmed the original work in that the hysteresis effects during unloading and reloading were sufficiently small to be neglected without a great loss in accuracy. Thus linear elastic behaviour may be assumed and, as this was shown to be independent of the value of the difference between maximum and minimum principal stresses, the elastic modulus could be represented as a function of the confining pressure only.

Tests carried out by the writers have not confirmed this final point in the pattern of behaviour, and it is of interest to compare and discuss the results obtained.

3. Experimental results

An experimental programme was carried out in order to investigate the general behaviour of a sample of dry sand under a varying stress field, with particular reference to the unloading and reloading characteristics. A full description of the experimental apparatus used for these tests may be found in [3]. All three principal stresses were independently varied as required, two being held constant and one varying during each experiment, by using a true tri-axial test. A further, more detailed discussion of this point of procedure is available [12]. In cases where the lateral stresses were equal, the sample behaviour was verified by duplicating the experiment using a standard tri-axial test.

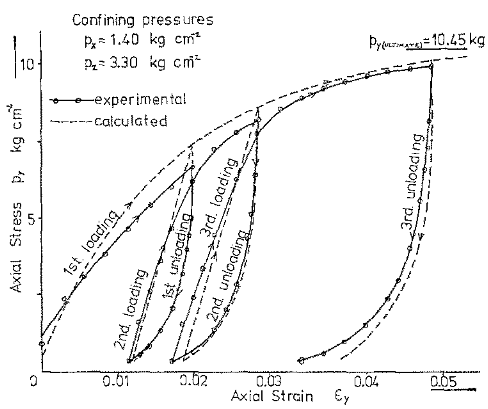

Figure 1 Calculated and experimental stress–strain curves for sand.

Representative results are shown in Fig. 1. The behaviour of the granular media subjected to first loading was found to be consistent with that represented by Daniel's equations [2]. The first loading behaviour was also accurately represented by the following incremental stress–strain relationships:

$$\delta\epsilon_x = \frac{\delta p_x}{K_1\left[\dfrac{\sqrt{(p_y p_z)}}{K_r} - p_x\right]} - \frac{\delta p_y}{K_2\dfrac{p_x}{p_z}\left[\dfrac{\sqrt{(p_x p_z)}}{K_r} - p_y\right]} - \frac{\delta p_z}{K_2\dfrac{p_x}{p_y}\left[\dfrac{\sqrt{(p_x p_y)}}{K_r} - p_z\right]} \tag{3a}$$

$$\delta\epsilon_y = -\frac{\delta p_x}{K_2\dfrac{p_y}{p_z}\left[\dfrac{\sqrt{(p_y p_z)}}{K_r} - p_x\right]} + \frac{\delta p_y}{K_1\left[\dfrac{\sqrt{(p_x p_z)}}{K_r} - p_y\right]} - \frac{\delta p_z}{K_2\dfrac{p_y}{p_x}\left[\dfrac{\sqrt{(p_x p_y)}}{K_r} - p_z\right]} \tag{3b}$$

$$\delta\epsilon_z = -\frac{\delta p_x}{K_2\dfrac{p_z}{p_y}\left[\dfrac{\sqrt{(p_y p_z)}}{K_r} - p_x\right]} - \frac{\delta p_y}{K_2\dfrac{p_z}{p_x}\left[\dfrac{\sqrt{(p_x p_z)}}{K_r} - p_y\right]} + \frac{\delta p_z}{K_1\left[\dfrac{\sqrt{(p_x p_y)}}{K_r} - p_z\right]} \tag{3c}$$

for which $p_x \leqslant p_y \leqslant p_z$

where K_1 ($= -K$) and K_2 are constants found experimentally. Comparison with Equations 2a to c shows that each leading diagonal term, which is of primary importance, is identical. The second order terms give substantially the same results numerically, but Equations 3a to c are considered more consistent.

Examination of Fig. 1 also shows that, at lower values of strain relief, the hysteresis loop due to unloading/reloading might be considered comparatively small, and that, at this stage, the stress–strain behaviour of the sample could perhaps, as a first approximation, be represented by a straight line. However, as cycling continues, at higher strain relief the hysteresis loop becomes larger, presumably due to greater slip at the particle interfaces. It is now clear that only the reloading portion of the loop can still be approximated by a straight line. This is clearly not acceptable for the unloading portion, because a significant error would be introduced.

An important feature to emerge is that, contrary to the findings of some previous researchers [2], the unloading/reloading relationship is not only a function of the confining pressure, but also depends on the stress levels during the cycling process, i.e. the level at which cycling is commenced and that at which reloading takes place.

The micro-behaviour of the granular specimen is undoubtedly complex, due to the different types of strain that take place during loading. Besides compaction of the grains themselves, slipping and rigid body movement of the grains takes place, and this behaviour depends on the nature and packing of the material. On unloading it is likely that the particle compaction is quickly relieved due to the small strains in the grains, and nearly all the rebound is made up by counter slips and free body movements. Again, during reloading some slipping will take place as well as grain compaction. Therefore, it is evident that prediction of the stress–strain behaviour of a granular material, such as sand, under complex loading is a very difficult problem and the best approach is clearly to make no attempt to isolate the different micro-strain components. It would also seem reasonable to assume linear reloading characteristics. A more thorough approach than this may well be an unnecessary refinement considering the nature of the material concerned.

With this object in mind, the following incremental equations were derived which satisfactorily predicted the behaviour of the specimen during the unloading sequence:

$$\delta\epsilon_x = \frac{\exp[-r_1 p_x/\sqrt{(p_y p_z)}]}{f_1\sqrt{(p_y p_z)}}\delta p_x - \frac{\exp[-r_2 p_y/\sqrt{(p_x p_z)}]}{f_2(p_z/p_x)^g\sqrt{(p_x p_z)}}\delta p_y - \frac{\exp[-r_3 p_z/\sqrt{(p_x p_y)}]}{f_2(p_y/p_x)^g\sqrt{(p_x p_y)}}\delta p_z \tag{4a}$$

$$\delta\epsilon_y = -\frac{\exp[-r_1 p_x/\sqrt{(p_y p_z)}]}{f_2(p_z/p_y)^g\sqrt{(p_y p_z)}}\delta p_x + \frac{\exp[-r_2 p_y/\sqrt{(p_x p_z)}]}{f_1\sqrt{(p_x p_z)}}\delta p_y - \frac{\exp[-r_3 p_z/\sqrt{(p_x p_y)}]}{f_2(p_x/p_y)^g\sqrt{(p_x p_y)}}\delta p_z \tag{4b}$$

$$\delta\epsilon_z = -\frac{\exp[-r_1 p_x/\sqrt{(p_y p_z)}]}{f_2(p_y/p_z)^g\sqrt{(p_y p_z)}}\delta p_x - \frac{\exp[-r_2 p_y/\sqrt{(p_x p_z)}]}{f_2(p_x/p_z)^g\sqrt{(p_x p_z)}}\delta p_y + \frac{\exp[-r_3 p_z/\sqrt{(p_x p_y)}]}{f_1\sqrt{(p_x p_y)}}\delta p_z \tag{4c}$$

for which $p_x \leqslant p_y \leqslant p_z$.

Where r_1, r_2 and r_3 are stress ratios defined by p_x (ultimate)/$p_{x\mathrm{i}}$, p_y (ultimate)/$p_{y\mathrm{i}}$ and p_z (ultimate)/$p_{z\mathrm{i}}$ respectively, $p_{x\mathrm{i}}$, $p_{y\mathrm{i}}$ and $p_{z\mathrm{i}}$ are the stress levels at which unloading commences, and f_1, f_2, g are constants found experimentally. The corresponding predictions are shown plotted in Fig. 1 for comparison with the experimental results.

As previously discussed, the reloading behaviour is represented as linearly elastic. This results in a small error at the junction with the main loading curve, but this is thought not to be excessive. Therefore, the reloading characteristics may be represented by the following equations:

$$\delta\epsilon_x = \frac{(\epsilon_{xx\mathrm{i}} - \epsilon_{xx\mathrm{f}})}{(p_{x\mathrm{i}} - p_{x\mathrm{f}})}\delta p_x - \frac{(\epsilon_{xy\mathrm{i}} - \epsilon_{xy\mathrm{f}})}{(p_{y\mathrm{i}} - p_{y\mathrm{f}})}\delta p_y - \frac{(\epsilon_{xz\mathrm{i}} - \epsilon_{xz\mathrm{f}})}{(p_{z\mathrm{i}} - p_{z\mathrm{f}})}\delta p_z \tag{5a}$$

$$\delta\epsilon_y = -\frac{(\epsilon_{yx\mathrm{i}} - \epsilon_{yx\mathrm{f}})}{(p_{x\mathrm{i}} - p_{x\mathrm{f}})}\delta p_x + \frac{(\epsilon_{yy\mathrm{i}} - \epsilon_{yy\mathrm{f}})}{(p_{y\mathrm{i}} - p_{y\mathrm{f}})}\delta p_y - \frac{(\epsilon_{yz\mathrm{i}} - \epsilon_{yz\mathrm{f}})}{(p_{z\mathrm{i}} - p_{z\mathrm{f}})}\delta p_z \tag{5b}$$

$$\delta\epsilon_z = -\frac{(\epsilon_{zx\mathrm{i}} - \epsilon_{zx\mathrm{f}})}{(p_{x\mathrm{i}} - p_{x\mathrm{f}})}\delta p_x - \frac{(\epsilon_{zy\mathrm{i}} - \epsilon_{zy\mathrm{f}})}{(p_{y\mathrm{i}} - p_{y\mathrm{f}})}\delta p_y + \frac{(\epsilon_{zz\mathrm{i}} - \epsilon_{zz\mathrm{f}})}{(p_{z\mathrm{i}} - p_{z\mathrm{f}})}\delta p_z \tag{5c}$$

where $\epsilon_{xy\mathrm{i}}$, $\epsilon_{xy\mathrm{f}}$, etc. are the strains in the x-direction due to loading in the y-direction at the commencement and end of the unloading portion respectively of the unloading/reloading cycle. These, too, are shown plotted in Fig. 1, for comparison with experiment.

It is now evident that the behaviour of a granular media under general loading conditions can be satisfactorily predicted using Equations 2, 3, 4 and 5 as soon as the constants for the particular sample have been found experimentally. The advantages over conventionally accepted approaches are clear and may be outlined as follows:

(1) the intermediate principal stress has been taken into account not only in the elasto-plastic behaviour but also in the rupture theory;

(2) attention has been paid to the unloading and reloading characteristics which have been shown to be rather more involved than previously proposed;

(3) the modulus of elasticity and Poisson's ratio, which are stress dependent under these circumstances, have been replaced by stress dependent coefficients.

4. Application to existing problems

There has been developing interest in the nonlinear behaviour of engineering soils [2, 4, 13] and for many engineering applications it is evident that a tractable non-linear representation of soil behaviour can provide useful results [2, 14, 15]. Some previous investigations [16] have tended to utilize existing soil theories and generally apply them to idealized problems which for more practical cases may not be a satisfactory description of soil behaviour. Others have approximated non-linear stress–strain behaviour either by iteration or incrementally, by repeated revision of the elastic properties during the stressing process. Both procedures have their short-comings [2]. The type of problem for which non-linear stress–strain theory has relevance includes the bearing capacity of foundations, cavitation and the ultimate load capacity of soil anchorage systems. Besides these applications having relevance in the field of civil engineering, they are also of increasing importance in the realm of military engineering, and marine exploration with associated resource utilization.

As mentioned in the introduction, other areas of study in which the proposed formulation may have application include investigations into the dry pressing of powdered materials. It may, therefore, be of importance to both ceramists and metallurgists.

It is of interest to note that the equations presented in this paper have been used in calculations carried out at Queen Mary College, London, to predict the ultimate capacities of both small and full scale anchors in order to complement the work already carried out in this area of study [17]. Work is proceeding in order to assess the anchor behaviour at working load and this is a particular area in which the relationships presented will be of importance.

As it is instructive to have reference to an example utilizing and manipulating incremental equations of the type presented, Appendix 1 outlines a cavitational theory incorporating these equations which is applicable to a semi-infinite mass of granular soil.

5. Conclusions

The aim of this paper was to prepare an approach for predicting the behaviour of granular media under a general loading cycle, including unloading and reloading. In accordance with the general three-dimensional stress system, the failure criteria is also dependent on the intermediate principal stress and not only the major and minor principal stresses which has been the case in conventional theory.

It is apparent that, for many practical cases of soil behaviour, a general stress–strain theory incorporating elasto-plastic concepts which can be determined using routine "true" tri-axial tests has some importance in the realm of engineering soils. Particularly as experience indicates that sands, unlike clays, have stress–strain characteristics which, for the same material, do not differ significantly whether in place or in the laboratory.

An interesting feature which may well be applicable to a large range of naturally occurring granular soils, even though due to their mode of deposition they may be anisotropic in their behaviour, is that when subjected to unloading and reloading, the reloading produces a nearly linear response. This suggests that it could be an indication of the behaviour of the soil when subjected to structural loading. Furthermore, if the structural loading is not excessive subsequent response could also be approximately linear.

Whereas this may in some cases be true for spread footings, other applications resulting in higher and more localized loading will probably require the concise statement of non-linearity, inelasticity and stress dependency in order to make reliable load/displacement predictions.

Appendix 1

Consider the expansion, by internal pressure, of a cylindrical cavity with axis vertical, located at a vertical depth H from the free surface of a semi-infinite granular soil mass; the sectional plan of the cavity is as shown in Fig. 2. Initially the cavity is of radius R_0 and, as the pressure is increased, the radius of the cavity will increase, with the soil in the annular zone adjacent to the cavity eventually attaining a state of limit equilibrium. Outside this annular zone the soil behaviour will be assumed elastic.

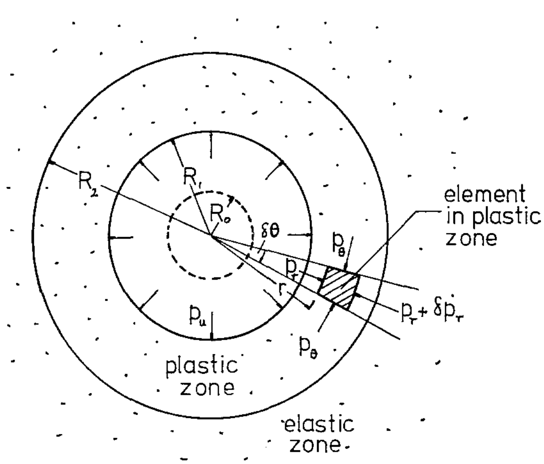

Figure 2 Sectional plan of cavity.

When the radius of the cavity is R_1 let the ultimate internal pressure be p_u, the radius of the plastic zone R_2 and the radial deformation of the surrounding elastic zone w.

Calculation of the ultimate pressure p_u

Considering the equilibrium of a soil element in the plastic, or elastic, zone results in the equation

$$\frac{\partial p_r}{\partial r} + \frac{(p_r - p_\theta)}{r} = 0 \cdot \qquad \text{(A1)}$$

The condition of rupture applicable to the plastic zone of a granular medium is, according to Daniel [3]:

$$p_r = \frac{1}{K_r}\sqrt{(p_y p_\theta)}$$

where $K_r = (1 - \sin\phi)/(1 + \sin\phi)$, i.e. the Rankine ratio. As the depth of embedment is H, it follows that

$$K_r p_r = \sqrt{(p_\theta \gamma H)} \qquad \text{(A2)}$$

where γ is the density of the soil.

Combining Equations A1 and A2 and solving the resulting differential equation, ensuring stress compatibility at the interface of the elastic and plastic zones; i.e.

$$\left.\begin{aligned} p_r + p_\theta &= 2K_0\gamma H \\ K_r p_r &= \sqrt{(p_\theta \gamma H)} \end{aligned}\right\}; \text{ at } r = R_2; \qquad \text{(A3)}$$

where K_0 = coefficient of earth pressure at rest, leads to:

$$p_r = \left\{ \frac{r}{R_2}\left[\frac{1}{-\dfrac{\gamma H}{2K_r^2} + \dfrac{\gamma H}{K_r 2}\sqrt{\left(\dfrac{1}{K_r^2} + 8K_0\right)}} - \frac{K_r^2}{\gamma H}\right] + \frac{K_r^2}{\gamma H}\right\}^{-1} \cdot$$

Furthermore, let $p_r = p_u$; at $r = R_1$;

$$P_u = \left\{ \frac{R_1}{R_2}\left[\frac{1}{-\dfrac{\gamma H}{2K_r^2} + \dfrac{\gamma H}{2K_r}\sqrt{\left(\dfrac{1}{K_r^2} + 8K_0\right)}} - \frac{K_r^2}{\gamma H}\right] + \frac{K_r^2}{\gamma H}\right\}^{-1} \cdot \qquad \text{(A4)}$$

In a similar fashion, the stresses in the elastic zone can be found from Equation A1 together with the condition

$$p_r + p_\theta = 2K_0\gamma H$$

again ensuring stress compatibility given by Equation A3.

Thus, for the elastic zone:

$$p_r = K_0\gamma H - \left(\frac{R_2}{r}\right)^2 \left[K_0\gamma H + \frac{\gamma H}{2K_r^2} - \frac{\gamma H}{2K_r}\sqrt{\left(\frac{1}{K_r^2} + 8K_0\right)}\right] \qquad \text{(A5a)}$$

and

$$p_\theta = K_0\gamma H + \left(\frac{R_2}{r}\right)^2 \left[K_0\gamma H + \frac{\gamma H}{2K_r^2} - \frac{\gamma H}{2K_r}\sqrt{\left(\frac{1}{K_r^2} + 8K_0\right)}\right] \cdot \qquad \text{(A5b)}$$

The deformation w in the radial direction at the interface of the elastic and plastic zones is given by:

$$w = \frac{1}{E}\int_{R_2}^{\infty} [p_r - \nu(p_\theta + p_y)]\,\mathrm{d}r$$

Hence

$$w = R_2 \frac{(1 + \nu)}{E}$$

$$\left[- K_0\gamma H - \frac{\gamma H}{2K_r^2} + \frac{\gamma H}{2K_r} \sqrt{\left(\frac{1}{K_r^2} + 8K_0 \right)} \right] \tag{A6}$$

where ν = Poisson's ratio and E = modulus of elasticity.

The use of the stress–strain relationships applicable to ideal perfectly elastic solids may be thought inappropriate when applied to granular soils. A further discussion of this point may be found in Appendix 2.

Consideration of the soil volumes before and after the expansion of the cavity yields the following relationship:

$$\frac{R_1^2}{R_2^2} = \frac{1}{(1 + \delta - k)} \left(2 \frac{w}{R_2} + \delta \right) \tag{A7}$$

where δ = average volumetric strain in the plastic zone and $k = R_0^2/R_1^2$. Hence eliminating w between Equations A6 and A7 and substituting for R_1^2/R_2^2 in Equation A4:

$$P_u = \left\{ \sqrt{\left\{ \frac{1}{(1 + \delta - k)} \left[\frac{2(1 + \nu)}{E} (A - K_0\gamma H) + \delta \right] \right\}} \left(\frac{1}{A} - \frac{K_r^2}{\gamma H} \right) + \frac{K_r^2}{\gamma H} \right\}^{-1} \tag{A8}$$

where

$$A = - \frac{\gamma H}{2K_r^2} + \frac{\gamma H}{2K_r} \sqrt{\left(\frac{1}{K_r^2} + 8K_0 \right)}.$$

This is an expression for the ultimate pressure in the cavity and can be determined, provided the average volumetric strain in the plastic zone is calculable.

Assessment of average volumetric strain in the plastic zone

It is clear from the rupture surface relationship that this surface in three-dimensional space is a conicoid. However, in this problem we are interested in its intersection with the p_y-constant plane, which is shown in Fig. 3.

Examination of Fig. 3 shows that point A is the state of stress in the soil before cavitation takes place. The shape of the rupture surface is clearly defined for a given value of p_y, but the extent of this surface, i.e. the extent of the plastic zone, is not specified. However, supposing that p_u were known, the plastic zone is defined and

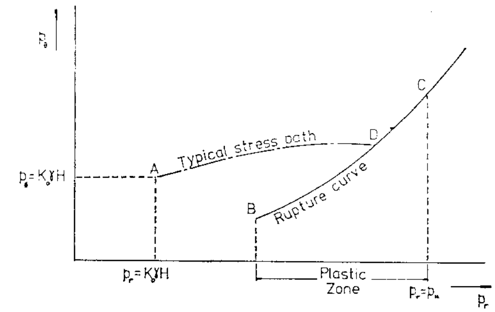

Figure 3 Stresses on p_y-constant plane.

the average volumetric strain in this region may be calculated from

$$\delta = \frac{\sum_{r=R_1}^{r=R_2} \delta_r \, dV}{\sum_{r=R_1}^{r=R_2} dV}$$

where δ_r is the volumetric strain for each increment of volume, e.g. the volumetric strain determined from a typical stress path AD shown in Fig. 2. Hence, specifying the step length in the summation process as t, we may re-write

$$\delta = \frac{\sum_{r=R_1}^{r=R_2} \delta_r 2\pi r t}{\pi(R_2^2 - R_1^2)} = \frac{2t \sum_{r=R_1}^{r=R_2} r\delta_r}{(R_2^2 - R_1^2)} \cdot \tag{A9}$$

The assessment of the strain is made using sets of equations such as Equation 2a to c.

As the material in the plastic zone has non-linear stress–strain characteristics, the principle of superposition is only valid for small increments of strain. Hence, equations such as Equations 2a to c must be applied incrementally along all stress paths from the starting point A to each point such as D on the rupture surface, with the volumetric strain being computed at each increment. Using this process, it is possible to compute the average volumetric strain for a given plastic zone.

In general, the computed volumetric strain will not be the same as the assumed value and, therefore, it is necessary to repeat the computation using revised values until convergence is achieved. Finally, stress step lengths are successively decreased until a sufficient degree of accuracy is obtained.

Conclusions

Summarizing the previous work, it can be seen that the calculation of the ultimate pressure causing cavitation may be carried out using the following iterative procedure.

(1) Initially, assume a value for the average volumetric strain in the plastic zone. Using Equations A6 to A8, calculate the radius of the plastic zone and also the ultimate pressure of the cavity.

(2) Using Equations A9 and 2a to c, calculate the average volumetric strain involved in the stress paths defined by Fig. 3. This calculated value should be the same as the initially assumed value. If not, repeat the iteration using a revised value of δ, and continue the process until the average volumetric strain is substantially the same as the assumed value.

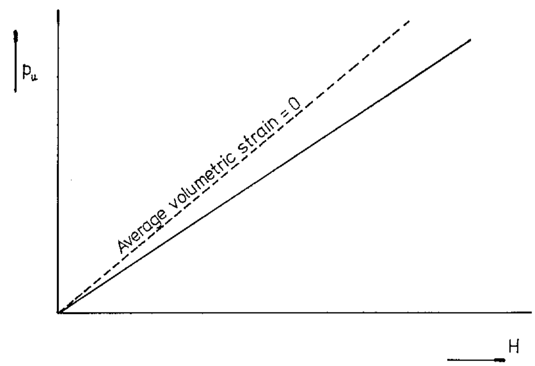

Figure 4 Variation of ultimate pressure with depth.

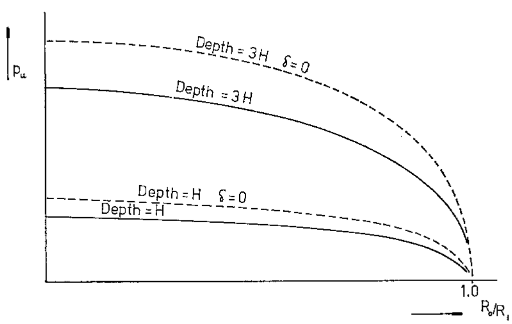

Figure 5 Variation of ultimate pressure with R_0/R_1.

Representative results of this computation are shown in Figs. 4 and 5. Fig. 4 shows the variation of the ultimate pressure with the depth of embedment for a typical granular soil. Also plotted are the results that would be obtained were the volumetric strain in the plastic zone adjacent to the cavity ignored. These results show a difference in the ultimate pressure of approximately 25% between the two cases. Fig. 5 shows the variation of the ultimate pressure with the ratio of initial to final size of cavity.

Appendix 2

It is significant that most writers in this field have assumed that the soil beyond the region of the plastic zone behaves as a perfectly elastic solid possessing a constant elastic modulus and Poisson's ratio. It is evident, however, that this assumption is worthy of further investigation to see if significant error results from its use.

Referring back to Equation 2a to c, in the case of $\delta\epsilon_x = \delta\epsilon_z = 0$, it is clear from symmetry that $p_x = p_z$. Hence equating $\delta\epsilon_x$ to zero and putting $p_x = p_z = K_0 p_y$, the following equation is obtained:

$$\frac{1}{K\left(1 - \dfrac{1}{K_r\sqrt{K_0}}\right)} - (2 - K_0)$$

$$\left[\frac{m}{\dfrac{1}{K_r} - 1} - \frac{n}{\dfrac{1}{K_r}}\right] = \left[\frac{m}{\dfrac{K_0}{K_r} - 1} - \frac{n}{\dfrac{K_0}{K_r}}\right]$$

Using the known constants for the material used, i.e. $K = -100$; $m = n = 0.009$; $\phi = 42°$; yields $K_0 \simeq 0.64$; This is the coefficient of earth pressure at rest.

Using equations such as Equations 2a to c and working along the stress paths defined by the initial state of the soil and Equations A5a and b, the radial deformation of the zone of the soil between $r = R_2$ and $r = \infty$ was computed. Examination of the incremental process proved that the behaviour of the soil in this zone was not markedly non-linear. The resulting apparent deformation characteristics were used to calculate depth-dependent values of the modulus of elasticity which, when used in Equation A6, resulted in values sufficiently close to those originally found by Equations 2a to c to render the use of the more involved approach, as outlined in this Appendix, an unnecessary refinement when calculating the ultimate pressure causing cavitation.

However, it is important to note that although careful choice of the elastic constants in this case simulated the elastic zone conditions, in general the modulus of elasticity and Poisson's ratio are stress-dependent, and hence functions of the stress paths.

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